Int. J. Solids Structures, 1970, Vol. 6, pp. 1185 to 1191. Pergamon Press. Printed in Great Britain

AN ISOTROPIC FRICTIONAL THEORY FOR A GRANULAR MEDIUM WITH OR WITHOUT COHESION†

J. L. DAIS

Institute of Technology, University of Minnesota, Minneapolis, Minnesota 55455

Abstract—A theory of de Jong [1] is modified and generalized to three dimensions. The corresponding global results are established which restrict orientations of surfaces of velocity discontinuity and surfaces separating rigid from deforming zones to principal stress directions. Previously unknown axi-symmetric and plane modes for the compression of a cylinder are found.

INTRODUCTION

IN AN idealization of a granular medium as a frictional material, deformation occurs by the smooth sliding of adjacent surfaces of material points. The idealization may or may not include cohesion. If the material is isotropic, the condition for frictional sliding is independent of the orientation of the surface of sliding. Consider an arbitrary plane through a point (the term "plane" is used in the local sense as the tangent plane to a surface), and let τ and σ_N denote respectively the magnitude of the shear component of surface traction and the normal component of surface traction on the plane. Tension is taken positive. Let c and $\tan\varphi$ denote respectively the cohesion and coefficient of friction between adjacent surfaces of material points along the plane. If the well-known Coulomb limit condition for the shear strength of soil is not reached, that is if $\tau < c-\sigma_N \tan\varphi$, relative sliding of the adjacent surfaces is not allowed to occur. If $\tau = c-\sigma_N \tan\varphi$, the surface may slide in the relative sense of the shear component of surface traction. τ cannot exceed $c-\sigma_N \tan\varphi$. If the material is isotropic, these conditions must be satisfied on all planes through the point, with c and φ independent of orientation.

Figure 1 is a pictorial representation of the frictional idealization. Figure 1(a) shows a homogeneous weightless block in a homogeneous state of stress, where $\sigma_1 > \sigma_2 > \sigma_3$ are principal stress components, shown as compressive. Suppose that for planes of all orientation the stress state is such that the Coulomb limit condition is not reached. Now, increase the magnitude of the compressive stress σ_3 until the Coulomb condition is just satisfied so that sliding or relative motion occurs on one of the planes perpendicular to the 1, 3 plane as shown in Fig. 1(b). Because of the symmetry, sliding could occur equally well on the complementary plane of Fig. 1(c). Sliding could also occur in discrete jumps on parallel planes,‡ as indicated in Figs. 1(d) and 1(e). In the limit, as the thickness of the elements in Figs. 1(d) and 1(e) approaches zero, sliding is smoothed out and can be regarded

† The conclusions presented in this report were derived in the course of research sponsored in part by the National Science Foundation under Grant GK1013 to Brown University and in part by the Division of Engineering, Brown University.

‡ It seems worth emphasizing the fact that this paper has nothing whatsoever to do with layered (and thus anisotropic) media. The inclined surfaces in Fig. 1 are determined solely by principal stress directions.

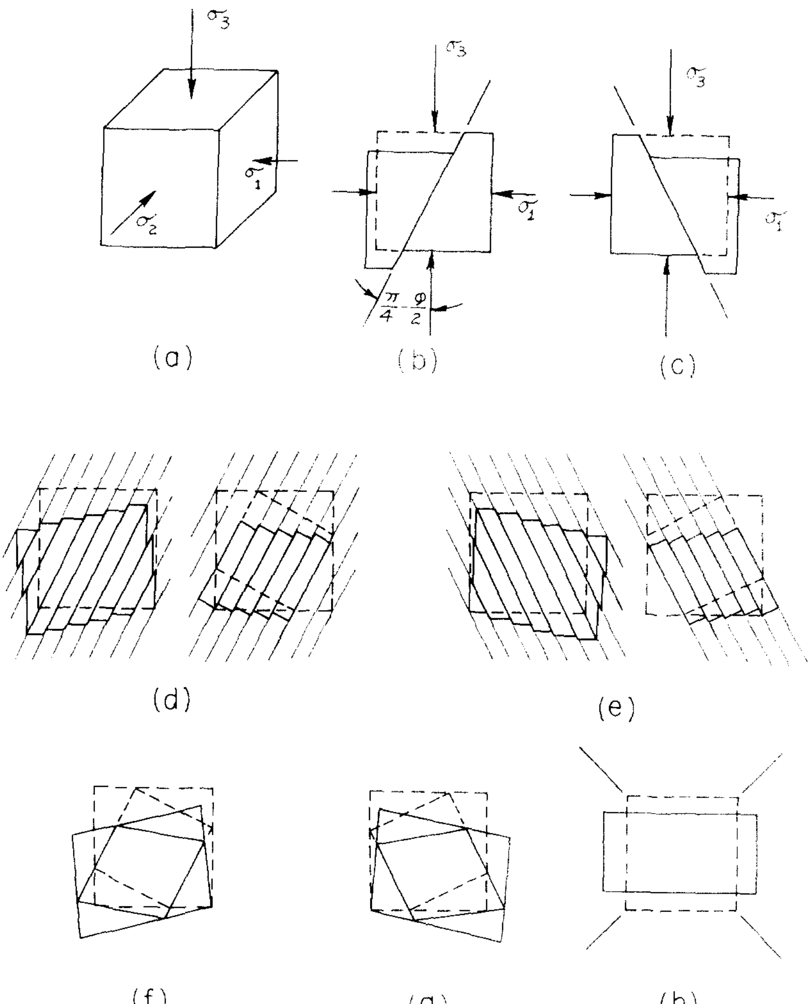

FIG. 1. Frictional material.

as occurring by a continuous shearing as in Figs. 1(f) and 1(g). Deformation may take place alternately in one continuous mode and then the other but the simultaneous occurrence of both continuous modes in Fig. 1(h) is not allowable in a frictional idealization. Such instantaneous combination does not occur by shearing or sliding on planes for which the Coulomb condition is met.

For such an ideal material, two global results can be obtained; the proof of these is a burden of this paper.

1. *A surface of velocity discontinuity has at each point as its tangent a surface upon which* $\tau = c - \sigma_N \tan \varphi$.

2. *A surface which separates a rigid zone from a deforming zone has at each point as its tangent either a surface upon which* $\tau = c - \sigma_N \tan \varphi$ *or a surface perpendicular to surfaces upon which* $\tau = c - \sigma_N \tan \varphi$. The first result has the widely known experimental versions

of cleavage surfaces in rocks† and failure surfaces in soils.‡ It has been widely observed in soil experiments that surfaces separating comparatively nearly rigid from rapidly deforming zones are reconcilable as having surfaces upon which $\tau = c - \sigma_N \tan \varphi$ as their tangents.§ It appears likely that a compression experiment on a short specimen|| would reveal the perpendiculars to surfaces upon which $\tau = c - \sigma_N \tan \varphi$ as separating such zones. The corresponding results in the theory of rigid perfectly plastic solids have played a prominent role in the analysis of metal plasticity problems; it seems reasonable that results 1 and 2 should be ingredients of mathematical formulations of problems for granular materials.

At this point it is worth discussing, from the standpoint of the formulation of problems, the implications of time averaging the continuous strain rate modes of Figs. 1(f) and 1(g). Within the framework of this theory, as in the theory of rigid perfectly plastic solids, one formulates problems for which the current configuration of a body is taken as the reference configuration. Part of a solution is then a velocity field which, if maintained for a while, changes the reference configuration to a neighboring configuration. The characterization of many real situations in which sizeable deformations take place is possible by alternate use of the modes of Figs. 1(f) and 1(g), but would require reposing a problem at intermediate configurations. This procedure can be avoided by pretending that the neighboring configuration has been reached directly by a velocity field for which these modes have been time averaged. Most probably the time averaging which will be of main analytical interest is that resulting in the coincidence of principal axes of stress and strain rate as in Fig. 1(h).¶ Result 2, which depends essentially on the restriction to the individual modes, can be retained provided that it is agreed that other modes are indeed attained by the successive action of the modes of Figs. 1(f) and 1(g).

It might at first appear that the frictional material could not be isotropic since the principal axes of stress and strain rate do not coincide. This issue has in fact already been raised by Spencer in reference to de Jong's theory. To date, coincidence has been established for only the following two types of ideal isotropic materials: (1) materials defined by a *unique* dependence of one tensor upon *only* one other tensor,†† and (2) rigid perfectly plastic materials‡‡ defined by the normality of the strain rate vector to a yield surface. Since the frictional material falls into neither of these classes, there is no inconsistency with material isotropy. The nonuniqueness in the constitutive relation for a frictional material has implications for nonuniqueness of collapse loads which is explored by Dais [8].

1. ALTERNATIVE APPROACHES AND PREVIOUS WORK

It might at first appear that an equivalent theory could be based on an assumed occurrence of deformation by a continuous shearing (with no volume change) upon planes on which $\tau = c - \sigma_N \tan \varphi$,§§ and then interpreting (as is done in the theory of rigid perfectly

† See, for example, Nadai [2] p. 330.

‡ See Jumikus [3] p. 498 and p. 507.

§ See Jumikus p. 498, or Sylvestrowicz [4] or Biarez *et al.* [5].

|| See Section 3 of the present paper.

¶ Consider the problem of a thick-walled cylinder loaded by external pressure and constrained between two fixed smooth rigid end plates. The axi-symmetric solution to this problem could easily be obtained through the use of the mode of Fig. 1(h).

†† See, for example, Serrin [6] pp. 230–232 for a proof.

‡‡ See Dais [7] for a proof.

§§ Geniev [9] developed a plane strain theory based on these assumptions.

plastic solids) a velocity discontinuity as the limiting case of rapid shearing in a thin zone. This approach leads one to the conclusion that surfaces perpendicular to surfaces upon which $\tau = c - \sigma_N \tan \varphi$ are allowable as surfaces of velocity discontinuity. As has been previously pointed out by Spencer [10], these surfaces have no such appropriate physical interpretation. The present interpretation of a continuous velocity field as the limiting case of a discontinuous field restricts discontinuities to occur only on surfaces on which $\tau = c - \sigma_N \tan \varphi$.

The deformation mechanism used in this paper was proposed by de Jong and the present theory can thus be regarded as an extension and modification of de Jong's work. De Jong considered this mechanism solely in the context of plane strain situations; in the present paper the mechanism is regarded as constituting the basis for a fully three dimensional theory. Results 1 and 2 are shown here to be inherent in the theory. De Jong allowed the superposition of the individual modes; the present approach does not object to a mathematically convenient time averaging, but suggests that phenomena consistent with the superposed modes but inconsistent with the successive action of the individual modes be disallowed.

For situations of equilibrium and plane stress, the well-known plane stress characteristic theory† fits into the structure of the present theory, and result 1 then becomes: *A line of velocity discontinuity has at each point as its tangent a stress characteristic.* Should, further, only the individual modes of Figs. 1(f) and 1(g) be considered, then the velocity characteristic theory of Geniev fits into the present structure and result 2 becomes: *A line which separates a rigid zone from a deforming zone has at each point as its tangent a velocity characteristic.* Geniev has exhibited velocity fields corresponding to straight line and log spiral slip line fields.

Spencer proposed a theory based also on the idea of shearing on a surface on which $\tau = c - \sigma_N \tan \varphi$. That theory differs from the present, though, in that Spencer postulated that the motion relative to a frame of reference which rotates with principal stress directions consists of superimposed shears. This leads Spencer to the conclusion that deformation occurs with no volume change and that strain rate depends upon stress rate as well as upon stress. Spencer neither postulates nor derives restrictions on the relative orientation of surfaces of shearing and surfaces upon which $\tau = c - \sigma_N \tan \varphi$. It appears to be not inconsistent with this theory for the rate of work done on the material to be negative.

2. CONSTITUTIVE RELATION

In the following, a plane on which $\tau = c - \sigma_N \tan \varphi$ holds is said to be a *slip plane*. The direction of the shear stress vector on the plane is said to be the *slip direction*. If there is a slip plane, then the stress state is said to be *critical*.

Let $\sigma_1 \geq \sigma_2 \geq \sigma_3$ be principal stress components and consider a critical state of stress. If $\sigma_1 > \sigma_2 > \sigma_3$ as in Fig. 1, there are two slip planes and the normals to these planes lie in the σ_1, σ_3 plane and are inclined at $+(\pi/4 - \varphi/2)$ and $-(\pi/4 - \varphi/2)$ to the direction of σ_1. The slip directions lie in the σ_1, σ_3 plane also. If $\sigma_2 = \sigma_3$, there are an infinity of slip planes with normals inclined at $(\pi/4 - \varphi/2)$ to the direction of σ_1. The slip directions lie

† See, for example, Sokolovskii [11].

in the plane defined by the direction of σ_1 and the normal to the slip plane. If $\sigma_1 = \sigma_2$, there are an infinity of slip planes with normals inclined at $(\pi/4 + \varphi/2)$ to the direction of σ_3. The slip directions lie in the plane defined by the direction of σ_3 and the normal to the slip plane.

If deformation occurs by sliding in a frictional material as in Figs. 1(b)–(e), then the velocity vector $\mathbf{v}$ experiences a jump, denoted as $[\mathbf{v}]$, across a slip plane and tangential to the slip direction on the plane. If $\dot{\varepsilon}_1 \geq \dot{\varepsilon}_2 \geq \dot{\varepsilon}_3$ denote principal strain rate components and sliding is smoothed out as in Figs. 1(f) and 1(g), then $\dot{\varepsilon}_2 = 0$ and deformation occurs with no volume change. A slip direction is then a direction of principal shear strain rate and of zero rate of extension. Consequently, if $\sigma_1 > \sigma_2 > \sigma_3$ as in Fig. 1, the direction of $\dot{\varepsilon}_1$ lies in the σ_1, σ_3 plane and is inclined at either $+\varphi/2$ or $-\varphi/2$ to the direction of σ_1. If $\sigma_2 = \sigma_3$, the direction of $\dot{\varepsilon}_1$ is inclined at $\varphi/2$ to the direction of σ_1. If $\sigma_1 = \sigma_2$, the direction of $\dot{\varepsilon}_1$ is inclined at $(\pi/2 - \varphi/2)$ to the direction of σ_3 and the direction of $\dot{\varepsilon}_2$ is perpendicular to the direction of σ_3.

Consider a surface in a region occupied by the frictional material. If this surface is to be a surface of velocity discontinuity, it must have everywhere as its tangent a slip direction, for otherwise $[\mathbf{v}]$ would not be tangent to a slip direction. If the surface is to separate a rigid zone from a deforming zone, then the surface must have everywhere as its tangent either a slip plane or a plane perpendicular to a slip plane. For, if the surface is a surface of velocity discontinuity, then it must be tangent to slip planes. Assume that the velocity is continuous across the surface but the surface is not tangent to slip planes or their perpendiculars. Then the element of the deforming zone immediately adjacent to the surface would be either extending or contracting which, for any nonzero length of surface, necessitates a discontinuity. This contradicts the hypothesis that the velocity is continuous across the surface, and the result is established.

3. COMPRESSION OF A CYLINDER

Consider the problem of compressing an initially right circular cylinder between two rough centrally loaded rigid platens which are free to shift laterally. Fig. 2(a) shows the slip planes and their perpendiculars corresponding to the homogeneous axially compressed stress state and Figs. 2(b) and 2(c) show some possible configurations following a small platen displacement. The conical regions of Fig. 2(c) are undeformed. In practice it is likely that the axi-symmetric modes will be associated with hardening stages of deformation and the plane modes with incipient softening. It is also likely that the modes (i) of Figs. 2(b) and 2(c) will be associated with cylinders of length/diameter ratio greater than $\tan(\pi/4 + \varphi/2)$ and modes (ii) with shorter cylinders. Modes (ii) are previously unknown.

The plane strained configurations are attained if a rate field with $\dot{\varepsilon}_1$ inclined as shown and homogeneous in the sheared zone is maintained during the platen displacement. The axi-symmetric configurations can be generated by using the plane modes as fundamental solutions by taking the shear zones to rotate through 360° about the vertical axis at constant angular speed as the platens are displaced at constant speed. It is straightforward but tedious to obtain equations for the deformed lateral surfaces. The axi-symmetric configurations so obtained are approximations to configurations which would be obtained by several successive equal plane shears at evenly distributed intervals in the 360°.

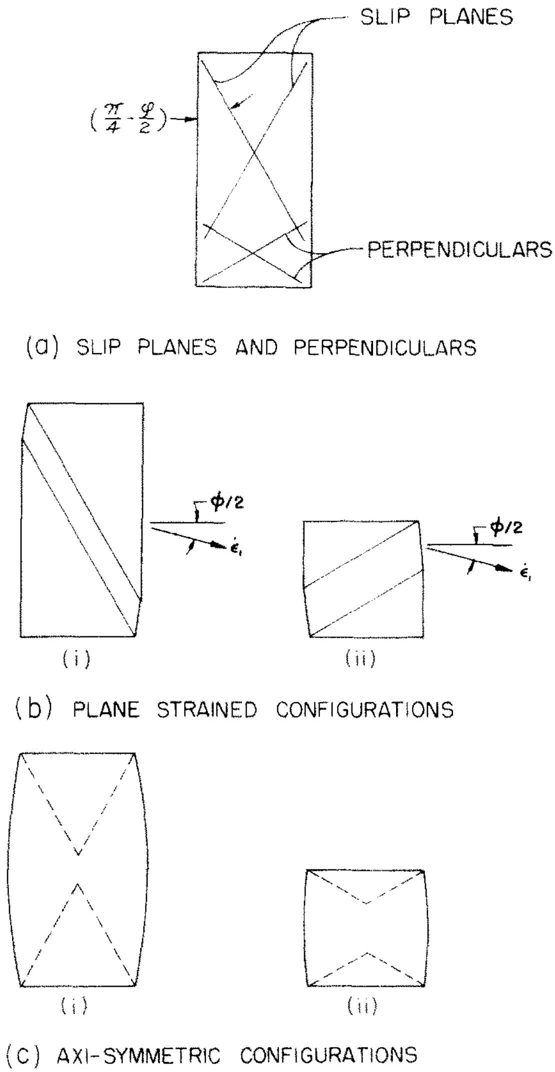

FIG. 2. Compression of initially right circular cylinder.

Acknowledgements—The author acknowledges Professor D. C. Drucker for guidance and criticism and Professor W. H. Warner for criticism.

REFERENCES

[1] G. DEJOSSELIN DEJONG, Statics and kinematics in the failable zone of a granular material. Uitgeverij, Delft.
[2] A. NADAI, *Theory of Flow and Fracture of Solids*, Vol. 1. McGraw-Hill (1950).
[3] A. R. JUMIKUS, *Soil Mechanics*. Van Nostrand (1962).
[4] W. SYLVESTROWICZ, Experimental investigation of the behavior of soil under a punch or footing. *J. Mech. Phys. Solids* **1**, 258–264 (1953).

[5] J. Biarez, M. Boucherie, L. M. Boucraut, J. Haeringer, D. Martin, B. Montel, R. Negre, P. Stutz et K. Wiendieck, Equilibre limite de «fondations» en milieu pulverulent a deux dimensions. *Rheologie et mécanique des sols*, *Proc. IUTAM Symp. Grenoble*, *April 1964*, edited by J. Kravtchenko and P. M. Sirieys, pp. 23–43. Springer (1966).
[6] J. Serrin, Mathematical principles of classical fluid mechanics. *Handbuch der Physik*, Bd. VIII/1, pp. 125–263. Springer (1959).
[7] J. Dais, On stress strain relations for isotropic rigid perfectly plastic solids. *Q. appl. Math.* **27**, 263–266 (1969).
[8] J. Dais, Nonuniqueness of collapse load for a frictional material. *Int. J. Solids Struct.* **6** (1970).
[9] G. A. Geniev, Questions on the dynamics of a granular medium. *Akademiia stroitel'stva i arkhitektury SSSR—Nauchnoe soobshchenie Tsentral'nye nauchno-issledovatel'skii institut stroitel'nykh konstruktsii* (1958); (available at the University of California, Berkeley).
[10] A. J. M. Spencer, A theory of the kinematics of ideal soils under plane strain conditions. *J. Mech. Phys. Solids* **12**, 337–351 (1964).
[11] V. V. Sokolovskii, *Statics of Granular Media* (translated by J. K. Lusher and A. W. T. Daniel). Pergamon (1965).

(*Received* 3 *March* 1969; *revised* 8 *December* 1969)

Абстракт—Модифицируется теория де Йонга и обобщается на три размера. Выводится соответствующие глобальные результаты, которые ограничивают направления поверхностей скорости непрерывности и поверхностей разделяющих жесткие зоны от деформированных зон в главных направлениях напряжений. Для цилиндра, даются ранее неизвестные осесимметрические и плоские режимы при сжатии.